Physics 367

Winter 2002

Problem Set 1

Solutions

1. a) simple cubic lattice

Let's assume that the lattice constant is a and that the spin at the origin is up: $S(0,0,0)=1$. Then its nearest neighbors are the points (x,y,z), where one of x,y,z is $\pm a$ and the others are zero. The spins at all these locations are equal to -1. The points closest to the origin where $S=1$ are $(\pm a\sqrt{2}, \pm a\sqrt{2}, 0)$, etc. Therefore, the sublattice of positive spins $\mathcal{L}_+$ is an FCC lattice of primitive vectors $\mathbf{e_1}=(a\sqrt{2}, a\sqrt{2}, 0), \mathbf{e_2}=(a\sqrt{2}, 0, a\sqrt{2}), \mathbf{e_3}=(0, a\sqrt{2}, a\sqrt{2})$ and conventional lattice constant $b=2a\sqrt{2}$. The sublattice $\mathcal{L}_-$ corresponding to spins $S=-1$ is identical.

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Problem Set 3

Solutions

9. Since $\beta\mathcal{H}_{\mathcal{E}}[\psi]$ is a polynomial in variables $\psi_1, \psi_2, \ldots, \psi_n$, it is equal to its Taylor expansion in the same variables, around any fixed value. Let us consider the Taylor expansion of $\beta\mathcal{H}_{\mathcal{E}}[\psi]$ around the point $(0,0,\ldots,0)$:

$$\beta\mathcal{H}_{\mathcal{E}}[\psi]=\sum_{m=0}^{\infty}\frac{1}{m}\left\{\sum_{\alpha_1,\ldots,\alpha_m=1}^{n}\tilde{U}_{\alpha_1,\ldots,\alpha_m}\psi_{\alpha_1}\cdot\psi_{\alpha_2}\cdot\ldots\cdot\psi_{\alpha_m}\right\}$$

Identifying this expansion with the closed form of $\beta\mathcal{H}_E[\psi]$, we conclude immediately that the only non-zero coefficients are $\tilde{U}_{ii}, \tilde{U}_{<iijj>}$ and $\tilde{U}_{<iijjkk>}$, where $i,j,k=\overline{1,n}$ and $<\ldots>$ means any permutation of the indices enclosed.

To determine the non-zero coefficients, we must consider all possible permutations of indices in the given combinations. Obviously, we have $\tilde{U}_{ii}=u_2, \tilde{U}_{iiii}=u_4, \tilde{U}_{iiiiii}=u_6$ for all $i=\overline{1,n}$, because there is no redundancy in these sets of indices. For the case $\tilde{U}_{<iijj>}$, $i\neq j$, we have an overall factor of $\frac{4!}{2!\cdot 2!}=6$, but the monoms $(\psi_i\psi_j)^2$ have an overall factor of 2 in $\beta\mathcal{H}_E[\psi]$ so we obtain $\tilde{U}_{<iijj>}=\frac{u_4}{3}$. For the case $\tilde{U}_{<iijjkk>}$, $i\neq j\neq k\neq i$, we obtain the combinatorial factor $\frac{6!}{(2!)^3}=60$, while the monoms $(\psi_i\psi_j\psi_k)^2$ are multiplied by $3!=6$ in $\beta\mathcal{H}_E[\psi]$, so $\tilde{U}_{<iijjkk>}=\frac{u_6}{10}$, if $i\neq j\neq k\neq i$. Finally, if $i\neq j$, we have a combinatorial factor of $\frac{6!}{4!\cdot 2!}=15$ and the monoms $(\psi_i)^4(\psi_j)^2$ have an overall factor of 3, thus $\tilde{U}_{<iiiijj>}=\frac{u_6}{5}$.

10. At the mean field configuration, $\beta\mathcal{H}_E[\psi]$ reaches a minimum. Since all the terms in the integral are positive, $\beta\mathcal{H}_E[\psi] \geq 0$ and we conclude that the minimum is obtained for $\psi_{MF}(x) = 0$ everywhere. To compute the Fourier transform of the correlation function, calculate the invers Green function

$$G^{-1}(x,y) = \frac{\delta[\beta\mathcal{H}_E]}{\delta\psi(x)\delta\psi(y)} = (r - c\Delta_x)\delta(x-y) + O(\psi^2)$$

The only term which might pose a problem in this derivation is the second one. Here is one way of deriving the correct sign:

Discretize $\beta\mathcal{H}_E[\psi]$ as

$$\beta\mathcal{H}_E[\psi] = \lim_{a\to 0, N\to\infty} \sum_{i=1}^{N} \left\{ \ldots + \frac{c}{2}\left[\frac{\psi_(ia+1) - \psi(ia)}{a}\right]^2 + \ldots \right\},$$

then

$$\frac{\partial[\beta\mathcal{H}_E]}{\partial\psi(ia)} = \lim_{a\to 0} c \left\{ \frac{\psi(ia) - \psi(ia+1) + \psi(ia) - \psi(ia-1)}{a^2} \right\} = -c\frac{\partial^2\psi}{\partial x^2}\delta(x - ia)$$

This derivation can be easily generalized to the multidimensional case.

Evaluating the invers Green function at the mean field configuration, we obtain

$$G_{MF}^{-1}(x,y) = (r - c\Delta_x)\delta(x-y),$$

so it is a diagonal operator. Its invers, the correlation function, will therefore be a diagonal operator as well:

$$G_{MF}(x-y) = (r - c\Delta_x)^{-1}$$

Using the Fourier transform for the correlation function (it is important to realize that the field vanishes along with *all* its derivatives at the mean field configuration), we can write

$$(r - c\Delta_x)\int \frac{d^dq}{(2\pi)^d} e^{iqx} G(q) = \delta(x-y) = \int \frac{d^dq}{(2\pi)^d} e^{iqx},$$

which gives $(r + cq^2)G(q) = 1$, so that

$$G(q) = \frac{1}{r + cq^2}.$$

11. By an argument similar to the one used in problem 10, we look for the minimum of $\beta\mathcal{H}_E[\psi]$ among the field configurations of *constant value*, $\psi(x) = \phi$. To find the minimum of $V(\phi)$, note that $V(\pm\infty) = \infty$ and that for $\phi < 0$, $V(\phi) > 0$. Moreover, $V(\phi) = 0 \Rightarrow \phi^2(\frac{1}{2} - \frac{v}{3}\phi + \frac{1}{4}\phi^2) = 0$. A simple calculation shows that if $v < \frac{3}{\sqrt{2}}$, the only real solution is $\phi = 0$, so the minimum of V is reached for $\phi = 0$. For $v = \frac{3}{\sqrt{2}}$, $V = 0$ both at $\phi = 0$ and at $\phi = \sqrt{2}$ and is positive for all other values. For the case $v > \frac{3}{\sqrt{2}}$, there are three solutions to $V(\phi) = 0$, namely $\phi = 0$ and two other, positive and distinct roots, ϕ_1 and ϕ_2. Clearly, in this case, V reaches its minimum at ϕ_c, somewhere in the interval $[\phi_1, \phi_2]$.

To summarize, we have:

For the case $v < \frac{3}{\sqrt{2}}$, the only solution is $\phi = 0$ and the problem reduces to problem 10. Thus, we obtain $\psi_{MF} = 0$ and $G_{MF}(q) = 1/(r + cq^2)$. All the higher order terms in V are irrelevant in this case.

If $v = \frac{3}{\sqrt{2}}$, the mean field configuration is degenerate, $\phi = 0$ and $\phi = \sqrt{2}$. The system will choose one of these minima depending on how the critical value $v_c = \frac{3}{\sqrt{2}}$ is approached. If v is approached from values smaller than v_c, then the mean field configuration is again $\phi = 0$. If v goes to $3/\sqrt{2}$ from larger values, the system will choose the configuration $\phi = \sqrt{2}$. In this case, one obtains for the invers Green function at $\phi = \sqrt{2}$

$$G^{-1}(x - y) = [-c\Delta_x + r(1 - 2\sqrt{2}v + 6)] = r - c\Delta_x,$$

so one retreives again the results of problem 10.

Finally, if $v > \frac{3}{\sqrt{2}}$, the minimum is reached when $\phi > 0$, $\frac{dV}{d\phi} = 0 \Rightarrow \phi[1 - v\phi + \phi^2] = 0 \Rightarrow [1 - v\phi + \phi^2] = 0$, and $\frac{d^2V}{d\phi^2} > 0 \Rightarrow [1 - v\phi + \phi^2] + \phi(2\phi - v) > 0 \Rightarrow \phi_c > \frac{v}{2}$. Eliminating the null root, the solutions to $\frac{dV}{d\phi} = 0$ are $\phi_\pm = \frac{v \pm \sqrt{v^2-4}}{2}$, so we conclude that $\phi_c = \frac{v+\sqrt{v^2-4}}{2}$. Evaluating $\left(\frac{d^2V}{d\phi^2}\right)_{\phi_c}$, we obtain

$$\left(\frac{d^2V}{d\phi^2}\right)_{\phi_c} = r\phi_c(2\phi_c - v) = \frac{(v + \sqrt{v^2 - 4})\sqrt{v^2 - 4}}{2},$$

and we label it r_c. After calculations identical to those in problem 10, we obtain

$$G(q) = \frac{1}{r_c + cq^2}.$$

12. Let us assume that $\alpha(b)$ is a monotonous, continuous function and define $f(x) = \log\alpha(\log b)$, where $x = \log b$ and $f = \log\alpha$. The function f will also be continuous and monotonous. Then the defining relation gives

$$\log\alpha(e^x) + \log\alpha(e^y) = \log\alpha(e^{x+y}) \Rightarrow f(x + y) = f(x) + f(y),$$

where $x = \log b$, $y = \log b'$. Note also that $\alpha(1) = 1$, so $\log\alpha(\log(1)) = 0 \Rightarrow f(0) = 0$. From the monotony of f, either $f(1) = 0$ and then $f(x) = 0$ for all $x > 0$, so $\alpha(b) = 1 = b^{f(1)}$, or $f(1) \neq 0$. We label $f(1) = y$. Consider now a positive integer q and write

$$y = f(1) = f\left(\frac{1}{q} + \ldots + \frac{1}{q}\right) = q \cdot f\left(\frac{1}{q}\right) \Rightarrow f\left(\frac{1}{q}\right) = \frac{y}{q}.$$

Let now p be any positive integer, then

$$\frac{p}{q} y = f\left(\frac{1}{q}\right) + \ldots + f\left(\frac{1}{q}\right) = f\left(\frac{p}{q}\right).$$

These relations prove that for any rational number $p/q \in \mathcal{Q}_+$, $f(\frac{p}{q}) = y\frac{p}{q}$. Since the set of rationals is dense in the set of reals and f is a continuous function, we obtain that for any sequence of rationals $p/q \to x \in \mathcal{R}_+$, $f(p/q) \to yx$. Thus, for any positive real x, $f(x) = yx \Rightarrow f(\log b) = y\log b \Rightarrow \alpha(b) = exp(f(x)) = b^y$.

13. From the defining formula for $M(\mu)$,

$$M(\mu) = \left\langle \frac{1}{\sqrt{V}} \int \frac{d^d q}{(2pi)^d} \psi(q) \right\rangle,$$

it follows that under the RG transformation,

$$\langle \psi(q) \rangle \to \alpha_b \langle \psi'(q) \rangle, \quad \sqrt{V} \to b^{d/2}\sqrt{V'}, \quad d^d q \to b^{-d} d^d q',$$

so we obtain the desired scaling law.

14. Let us define the integral $I(y) = \int dtj(t,y)\phi(t)$, where j is a suitably chosen distribution, independent on ϕ. Then it is straightforward to compute (for example, by discretization) that

$$\frac{\delta I(y)}{\delta \phi(x)} = j(y,x).$$

Consider now the exponential $e^{I(y)} = \sum_{n=0}^{\infty} \frac{I(y)^n}{n!}$ and compute

$$\frac{\delta e^{I(y)}}{\delta \phi(x)} = \left(\sum_{n=0}^{\infty} \frac{I(y)^n}{n!} \right) j(t,x).$$

Now make the choice of distribution $j(t,y) = \lambda\delta(t-y)$ and get $I(y) = \lambda\phi(y)$, so $e^{I(y)} = e^{\lambda\phi(y)}$ and

$$\frac{\delta e^{I(y)}}{\delta \phi(x)} = \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \frac{\delta^n \phi(y)}{\delta \phi(x)} \right) = \left(\sum_{n=0}^{\infty} \frac{(\lambda\phi(y))^n}{n!} \right) \lambda\delta(x-y)$$

Identifying the coefficients in the λ expansion gives us the desired result.

Next, consider the integral $J = -\frac{1}{2}\int dy\phi^2(y)$ and compute

$$\frac{\delta J}{\delta\phi(x)} = -\phi(x)$$

(again, by discretization). Repeating the calculation above we get

$$\frac{\delta e^J}{\delta\phi(x)} = -\left(\sum_{n=0}^{\infty}\frac{J^n}{n!}\right)\phi(x) = -\phi(x)e^J.$$

For the last excercise, rewrite $J = \int dy\phi^n(y)(\nabla\phi(y))^2$ as

$$\begin{aligned} J = \int dy\nabla\left(\frac{\phi^{n+1}(y)}{n+1}\right)\cdot\nabla\phi(y) = \\ \int dy\nabla\left(\frac{\phi^{n+1}(y)}{n+1}\nabla\phi(y)\right) - \int dy\frac{\phi^{n+1}(y)}{n+1}(\nabla^2\phi(y)) = \\ -\int dy\frac{\phi^{n+1}(y)}{n+1}(\nabla^2\phi(y)) \end{aligned} \tag{1}$$

for any well-behaved function ϕ.

Formal derivation then gives

$$\frac{\delta J}{\delta\phi(x)} = -\left[\phi^n(x)\nabla^2\phi(x) + \frac{\phi^{n+1}(x)}{n+1}\nabla_x^2\right]$$

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Problem Set 4

Solutions

15. Define the Fourier transform as

$$I(r,p) = \frac{1}{(2\pi)^d}\int_{-\infty}^{\infty} d^d\vec{q}\frac{e^{i\vec{q}\vec{r}}}{q^2+p^2},$$

where we identify $p = \xi^{-1}$. The fact that I only depends on the magnitude of $\vec{r}$ is easy to understand, given the explicit invariance of the integral under rotations of vectors $\vec{r}$ and $\vec{q}$. Now define the function

$$J(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dte^{itp}I(r,p),$$

in other words, the Fourier transform of I with respect to the variable p. Obviously,

$$I(r,p) = \int_{-\infty}^{\infty} dt e^{-itp} J(r,t),$$

so the problem becomes finding the function J. Applying the $d+1$ dimensional Laplace operator $\Delta_{d+1} = \Delta_{\vec{r}} + \frac{\partial^2}{\partial t^2}$ to J, we obtain by elementary calculus

$$\Delta_{d+1} J(r,t) = -\delta^{d+1}(\vec{r},t).$$

This important relation tells us that J is the Green's function of the Laplace operator in $d+1$ dimensions. In more physical terms, it is the electrostatic potential created by a negative unit charge sitting at the origin in the space $\mathcal{R}_{d+1}$.

The Green's function of the Laplace operator in $d+1$ dimensions is (see note 1 below for details)

$$G_{d+1}(\vec{x}) = \frac{1}{(1-d) S_{d+1} \left(\sqrt{x_1^2 + x_2^2 + \ldots + x_{d+1}^2}\right)^{d-1}},$$

where $d > 1$ and S_n is the area of the sphere of unit radius in n dimensions.

We conclude that

$$I(r,p) = \frac{1}{(d-1) S_{d+1}} \int_{-\infty}^{\infty} dt \frac{e^{-itp}}{\left(\sqrt{r^2 + t^2}\right)^{d-1}}.$$

To compute this integral, use a trick usually attributed to Feynman, but due to Lagrange. From the normalization formula of gaussian integrals, derive

$$\frac{1}{\sqrt{\alpha^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-\frac{y^2 \alpha^2}{2}},$$

so

$$I(r,p) = \frac{1}{(d-1)(\sqrt{2\pi})^{d-1} S_{d+1}} \int_{-\infty}^{\infty} dt dy_1 dy_2 \ldots dy_{d-1} e^{-itp - \frac{r^2+t^2}{2}\left[y_1^2 + \ldots + y_{d-1}^2\right]}$$

Making the change of dummy y coordinates from cartesian to spherical, we obtain

$$I(r,p) = \frac{S_{d-1}}{(d-1)(\sqrt{2\pi})^{d-1} S_{d+1}} \int_{-\infty}^{\infty} d\rho dt \rho^{d-2} e^{-itp - \frac{\rho^2(r^2+t^2)}{2}}.$$

Now perform the integral in the variable t by completing the square and applying the gaussian formula and obtain

$$-itp - \frac{t^2\rho^2}{2} = -\frac{\rho^2}{2}\left[t + \frac{ip}{\rho^2}\right]^2 - \frac{p^2}{2\rho^2}$$

$$I(r,p) = \frac{S_{d-1}}{(d-1)(\sqrt{2\pi})^{d-2} S_{d+1}} \int_{-\infty}^{\infty} d\rho \rho^{d-3} e^{-\frac{1}{2}\rho^2 r^2 - \frac{p^2}{2\rho^2}}.$$

The experienced reader will realize at this point the similarity between this integral and the so-called *Sonine* integral formula defining Bessel functions. To make the analogy work, perform the change of variables

$$\left[r^2\rho^2 + \frac{p^2}{\rho^2}\right] = -z\left[e^{i\zeta} - e^{-i\zeta}\right] \Rightarrow -z^2 = (rp)^2 \Rightarrow z = irp.$$

Also, from

$$r^2\rho^2 = irpe^{-i\zeta} \rightarrow \rho = \sqrt{\frac{p}{r}} e^{i\left[-\frac{\zeta}{2} + \frac{\pi}{4}\right]}$$

The same formula indicates that the new variable must have the form $\zeta = \frac{\pi}{2} + iw$, where w is any real number. Finally, $\rho = 0$ corresponds to $w = -\infty$.

We conclude that the integration now takes place along the line $Re(\zeta) = \frac{\pi}{2}$ and the integral becomes (from $\frac{d\rho}{\rho} = -\frac{i}{2} d\zeta$)

$$I(r,p) = \frac{S_{d-1}}{(d-1)(\sqrt{2\pi})^{d-2} S_{d+1}} \frac{i}{2} \left(\sqrt{\frac{p}{r}}\right)^{d-2} e^{\frac{i\lambda\pi}{2}} \int_{\mathcal{C}} d\zeta e^{iz\sin\zeta - i\lambda\zeta},$$

where $\lambda = \frac{d}{2} - 1$ and $\mathcal{C}$ is the contour described above. From the standard integral definitions of the Bessel functions of the third kind (the Hankel functions),

$$H_\lambda^{(1)}(z) = \frac{1}{\pi} \int_{\mathcal{C}} d\zeta e^{iz\sin\zeta - i\lambda\zeta},$$

so

$$I(r,p) = \frac{S_{d-1}}{(d-1)(\sqrt{2\pi})^{d-2} S_{d+1}} \left(\sqrt{\frac{p}{r}}\right)^{d-2} \frac{i\pi}{2} e^{\frac{i\lambda\pi}{2}} H_\lambda^{(1)}(irp).$$

The combination

$$\frac{i\pi}{2} e^{\frac{i\lambda\pi}{2}} H_\lambda^{(1)}(irp) = K_\lambda(rp)$$

is one of the Bessel functions of the fourth kind and was first identified by MacLauren (1899). We conclude that

$$I(r,p) = \frac{S_{d-1}}{(d-1) S_{d+1}} \left(\sqrt{\frac{p}{2\pi r}}\right)^{d-2} K_{\frac{d}{2}-1}(rp).$$

Using the formula (see note 2)

$$S_d = \frac{(\sqrt{2\pi})^d}{2^{d/2-1}\Gamma(d/2)},$$

we finally conclude that

$$I = \frac{1}{2\pi}\left(\sqrt{\frac{1}{2\pi\xi r}}\right)^{d-2} K_{\frac{d}{2}-1}(r/\xi).$$

For future reference, please note that

$$K_{1/2}(rp) = \sqrt{\frac{\pi}{2rp}}e^{-rp},$$

so for $d = 3$ we obtain

$$I = \frac{1}{2\pi}\sqrt{\frac{p}{2\pi r}}\sqrt{\frac{\pi}{2rp}}e^{-rp} = \frac{e^{-r/\xi}}{4\pi r},$$

the Yukawa formula for the correlation function due to interactions through a massive particle.

Note 1. If $n > 2$, the Green's function for the Laplace operator in n dimensions is

$$G_n(r) = \frac{1}{(2-n)S_n r^{n-2}}$$

simply because its gradient is

$$\vec{\nabla} G_n = \frac{\hat{r}}{S_n r^{n-1}},$$

so

$$\int_{r\le R} d\vec{r}\Delta G_n(r) = \int d\vec{r}\vec{\nabla}\cdot\vec{\nabla}G_n = \frac{\int_{r=R} d\Omega_n}{S_n} = 1,$$

for any value of the radius. We also have

$$\Delta_n = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\Delta_{angles}}{r^2},$$

which by brute force calculations gives

$$\Delta_n G_n = 0,$$

for all $r \neq 0$, so G_n is the Green's function for the Laplace operator in n dimensions.

Note 2 The standard method for finding the area of the sphere of unit radius in d dimensions is computing the integral

$$I = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \right)^d$$

in two different ways. First, use the gaussian normalization and obtain

$$I = (\sqrt{2\pi})^d.$$

Changing from cartesian to spherical coordinates, we obtain

$$I = S_d \int_0^\infty dr [r^{d-1} e^{-\frac{r^2}{2}}] = 2^{d/2-1} S_d \int_0^\infty dt t^{(d/2-1)} e^{-t} = 2^{(d/2-1)} S_d \Gamma(d/2).$$

Equating the two expressions, one gets

$$S_d = \frac{(\sqrt{2\pi})^d}{2^{(d/2-1)}\Gamma(d/2)}.$$

16. Since only the leading term in r is required, use

$$\left[1 + \frac{r}{ck^2}\right]^{-n} \simeq 1 - n\frac{r}{ck^2}$$

and obtain for the first integral

$$\frac{\tilde{K}_4}{2c}\left[\Lambda^2 - \left(\frac{\Lambda}{b}\right)^2\right] - \frac{\tilde{K}_4 r}{c^2} \log b.$$

The same approximation yields for the second integral,

$$\frac{2\tilde{K}_4}{c^2} \log b - \frac{2\tilde{K}_4 r}{c^3 \Lambda^2}(b^2 - 1).$$

17. The determinant is

$$\Delta = (b^{y_1} - \lambda)(b^{y_2} - \lambda),$$

so the eigenvalues are

$$\lambda_1 = b^{y_1}, \quad \lambda_2 = b^{y_2}.$$

The normalized eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \left(\sqrt{1 + \left[\frac{b^{y_2} - b^{y_1}}{D_0(b^2 - 1)}\right]^2} \right)^{-1} \begin{pmatrix} 1 \\ \frac{b^{y_2} - b^{y_1}}{D_0(b^2-1)} \end{pmatrix}.$$

18. As indicated in previous homework sets, the mean field configuration is constant, independent on x, so it is found by minimizing the polynomial in $|\psi|^2$.

Case 1. For $T > T_c$, all coefficients are positive, so $\psi_{MF} = 0$. In this case,

$$\frac{\delta^2[\beta\mathcal{H}_E]}{\delta\psi(x)\delta\psi(y)} = (|r| - c\Delta)\delta(x-y),$$

so

$$C(\vec{q}) = \frac{1}{|r| + cq^2},$$

Case 2. For $T < T_c$, the mean field configuration is given to first order by

$$-|r| + u|\psi|^2 + u^{(6)}|\psi|^4 = 0 \Rightarrow \psi_{MF} \simeq \sqrt{\frac{|r|}{u}},$$

because for $|r| \to 0$, a power expansion of ψ in $\sqrt{|r|}$ washes away the term proportional to ψ^4. The stability matrix will be given by

$$\frac{\delta^2[\beta\mathcal{H}_E]}{\delta\psi(x)\delta\psi(y)} = (-|r| - c\Delta + 3u\psi^2 + 5u^{(6)}\psi^4)\delta(x-y),$$

so that

$$C(\vec{q}) = \frac{1}{2|r| + cq^2 + O(|r|^2)}$$

We conclude that, close to T_c,

$$\psi \sim |T - T_c|^{1/2}, \quad C(\vec{q}) = \frac{1}{2|r| + cq^2} \Rightarrow \xi^{-2} \sim |T - T_c|, \quad C(q, T = T_c) \sim \frac{1}{q^2}, \quad C(0) \sim |T - T_c|^{-1},$$

so that

$$\beta = \frac{1}{2}, \quad \nu = \frac{1}{2}, \quad \eta = 0, \quad \gamma = 1,$$

and the coefficients do not depend on $u^{(6)}$.

19. The coefficient f must be positive, otherwise a saddle point (mean field) configuration may not exist. This is not allowed, because any physical system must have a ground state to be observable. The mean field configuration will therefore be given by a constant value ψ_{MF}, so it will be determined again by the value minimizing the potential in ψ. We conclude that the analysis performed in problem 18 applies and that the critical behavior does not depend on f. The only contribution this term could have is in the exponent η, but since for small q, q^4 is negligible compared to q^2, we obtain again $\eta = 0$.

To compute the variance $\langle(\delta x)^2\rangle$, note that

$$S = M(0) = \left(\frac{V}{(2\pi)^d}\right)^{1/2} \int d^d q \frac{1}{\kappa|r| + cq^2 + fq^4},$$

where κ is 1 for $T > T_c$ and 2 for $T < T_c$. Obviously, the integral does not diverge for small values of Λ (infrared divergence), and for large values of Λ it behaves as

$$\int^{\Lambda} \frac{q^{d-1}}{q^4} dq \sim \log \Lambda \text{ for } d = 4 \text{ and } \Lambda^{d-4} \text{ for } d \neq 4.$$

We conclude that the integral converges for $d < 4$. For $d = 3$, we have

$$I = \int_0^\infty dq \frac{q^2}{fq^4 + cq^2 + \kappa|r|}.$$

The polynomial $P(q) = fq^4 + cq^2 + \kappa|r|$ has roots given by

$$(q^2)_{1,2} =_{def} -a_{1,2}^2 = -\frac{-c \pm \sqrt{c^2 - 4\kappa|r|f}}{2f}.$$

We can use the decomposition

$$\frac{q^2}{(q^2 + a_1^2)(q^2 + a_2^2)} = \frac{1}{a_1^2 - a_2^2}\left[\frac{a_1^2}{q^2 + a_1^2} - \frac{a_2^2}{q^2 + a_2^2}\right]$$

and obtain

$$I = \int_0^\infty dq \frac{q^2}{f[(q^2 + a_1^2)(q^2 + a_2^2)]} = \frac{1}{f}\int_0^\infty dq \frac{1}{a_1^2 - a_2^2}\left[\frac{a_1^2}{q^2 + a_1^2} - \frac{a_2^2}{q^2 + a_2^2}\right]$$

Now use

$$\int_0^\infty dx \frac{1}{x^2 + a^2} = \frac{\pi}{2a}$$

and write

$$I = \frac{\pi}{2f}\frac{1}{a_1 + a_2} \simeq \frac{\pi}{2\sqrt{cf}} \text{ as } \; f \to 0,$$

where the positive constants $a_{1,2}$ have been previously defined. Clearly, the integral diverges if $f = 0$.